Pickwick's Umbrella

In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won't rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3. Find

(a) the probability that Pickwick has no umbrella, given that it rains.
(b) the probability it doesn't rain, given that he brings his umbrella.

The idea of solving this problem is to get a tree set up with the correct structure to pick off the joint probabilities of U and ~U conjoined with all the possibilities. The hardest part is to unravel the conditionals – we're given the probability that it rains given that rain is predicted -- we need the probability that she predicted rain given that it rains --- and after some thought* , we see that these are the same probabilities. I use "F" for forecasted (predicted) rain, "R" for rain, and "U" for umbrella, "~" for negation, and "." for conjunction.

So, P(R)=1/2, P(~R)=1/2, P(F/R)=P(~F/~R)=2/3, P(U/Fr)=1, P(U/~F)=1/3

And then for (a)
P(~U/R) = P(~U.R) / P(R) = 1/9 / 1/2 = 2/9

And for (b)
P(~R/U) = P(~R.U) / P(U) = (1/6 + 1/9) / (1/3 + 1/18 + 1/6 + 1/9) = 5/18 / 12/18 = 5/12

Use a tree to check that:

R.F.U = 1/3	~R.F.U = 1/6
R.F.~U = 0	~R.F.~U = 0
R.~F.U = 1/18	~R.~F.U = 1/9
R.~F.~U = 1/9	~R.~F.~U = 2/9

* You need to show that P(F) = 1/2. You can do this by using the two equations

a. P(R/F) * P(F) = P(F.R) = P(R) * P(F/R)

b. P(~F/R) * P(R) = P(~F.R) = P(R/~F) * P(~F)

Use the fact that P(~F/R) = 1 - P(F/R) and P(R/~F) = 1 - P(~R/~F) = 1 - 2/3 = 1/3

Then substituting the remaining known values, we get

a. 2/3 * P(F) = 1/2 * P(F/R)

b. (1- P(F/R)) * 1/2 = 1/3 * (1- P(F))

two equations in two unknowns, with solution P(F) = 1/2, P(F/R) = 2/3.

More generally, one can write the above two equations as

a. $a X = b Y$

b. $b (1 - Y) = (1 - a) (1 - X)$

Solving for X in terms of a & b yields $X = \frac{1 - (a + b)}{1 - 2a}$, c*

Because of the impossibility of certain pairs P(R|F) = P(~R|~F), P(R) c* are the constraints on a & b which make sure that $0 < X < 1$. (namely $a \neq ½$, when $a < ½$, $a < b$, $a + b < 1$ and when $a > ½$, $a > b$ and $a + b > 1$)

HINT: Show that for any two events R & F such that

a = P(R/F) = P(~R/~F)

$P(F) = \frac{1 - (a + b)}{1 - 2a}$ where b = P(R)

In the case where b = ½ , P(F) = ½ for any a. So, P(F) = ½ as long as P(R|F) = P(~R|~F), and P(R)= ½

Thinking of urns and looking into all the cases where P(F) = P(R)= ½ , if you have two urns, F & ~F, each containing N white balls and reds balls, and you draw from each with probability ½ , and you want (or find that) the probability of drawing a red ball is ½, then you know that if there are n red balls in F, there must be N-n red balls in ~F. That is, a = P(R/F) = P(~R/~F) can be any probability. The symmetry alone serves to balance the ½ probability of either red or not-red.