Situations and You

It is uncommon for math people to come out of their shells and try to talk about things that everyday folk can understand. Unfortunately, a limit to the amount of material that can be expressed without preliminary foundations which are not a part of everyday knowledge is quickly found by most authors.

One way to proceed then is to present preliminary foundational material, and then move on. Another way is to embed the preliminary foundational material within the exposition in such a manner that the reader is unaware it's not a part of the exposition. For instance, before talking about meaning and partiality - which is what I'd like to present here in Situations and You, preliminary required could be a good exposition on Situation Semantics that is a very rudimentary, and elementary and which, when understood, can lead the way to an understanding of formal systems and models that deal with natural languages and the (worlds) situations we encounter. I include the following here, but there are many others that are easily Google searched, if for some reason you don't like this one, or need more: HHL_SituationTheory.pdf

As I've said, another way of exposing knowledge is to forgo the preliminaries and incorporate them within the exposition. Since all cannot be forgone, documents on semantics, language, logic, knowledge, and learning should be used as reference and particular ones will be not only be referenced by also provided by hyperlinks.

Hawaii,1975

In 1975 I lived in Honolulu where I attended the University of Hawaii. I learned well the use of chop sticks, because at the fast food restaurants, that's all you had to scope up your rice. I saw roaches as big as small cats, and had a house lizard that ate the small ones unfortunate enough to come in to my apartment. I lived about a kilometer away from the University of Hawaii at Manoa, in Honolulu, and walked along Wilder Avenue to get there. Since it rains on and off in Hawaii, generally a light pleasant rain that refreshes the outdoor temperatures of around twenty five degrees Celsius, I often was drenched by the time I arrived for classes. It wasn't that I forgot my umbrella, it was that often it would rain without there being a cloud in the sky, as the fierce winds well above the island would carry the rain miles away from its cloud source. This was the world I lived in, quite different from my hometown of San Francisco, where not so many anomalies were found. The rain without clouds, chop sticks at fast food restaurants, and house lizards were all believable but in most cases not what many would think these as being part of their normal world. In my Hawaiian world, not carrying an umbrella, not eating with chop sticks, and shooing out house lizards are all mistakes; and to live successfully in that world, one would want to do just the opposite. It should be easy to see that there is a distinction between our action-events and the world that we live in since our action-events are somewhat dictated by the world we live in. If you could write down all the things that you should do in a particular world to be successful and happy in it, and give the list of those things to someone that knows nothing about you, that someone would be able to understand the world you live in. The communication can take the form of you listing all the things you do in that world to make you happy. Another way to tell the person about your world is that you say "I live in Hawaii". The other person might be knowledgeable about Hawaii, and might be able to tell you things on the list you've made. If the person could do so – tell you how best to get along in your world, then you might think the person as having Knowledge about your world. Just how the person obtained the knowledge may not be known. Perhaps he/she is a psychic, or perhaps he/she lived in Hawaii at one time, or perhaps he/she has acquired knowledge by reading, or from a friend, or perhaps from living in or hearing about a place similar and making a guess. You have learned how to live in your world, and your find learned that as well. So there is Learning, an acquisition of knowledge that can take many forms, and what is communicated is done so in a Language, and what is talked about are actualities, or Worlds, in this case tangible things that exist. These four ideas are separate, yet together they make up the idea of Knowledge Representation, and with Logic thrown in, we can get Reasoning.

Learning

Language World

Logic

Bill, Peggy and Catherine

My great friend Bill was a native, called a Kamaiina, for old timer. He didn't mind roaches, so he had no house lizards. A girlfriend, Peggy, from Texas, lived near Waikiki Beach, in a high rise, and wasn't aware of any cock-roach problems; but she did have to use chop sticks and did get caught in the rain without clouds occasionally. The three of us were graduate math students at UH. I was a sort of black sheep because I was actually studying mathematical logic. Peggy never went to Bill's house, but the three of us often ate fast-food lunch together using chopsticks. I guess you could say we were three different types of people caught up in the same world, 1975 Hawaii. This world had a definite geographic nature, thus the rain without clouds, a definite thus cultural nature, the fast-food served with chopsticks, and a definite cleanliness nature, thus cock-roaches. Peggy was totally oblivious to the cleanliness nature since the immediate surroundings she lived in were clean and void of the pests. Bill was totally oblivious to a world were fast-food wasn't served up with chopsticks. Each of us knew of different worlds, and each of us had are likes and dislikes in our current world; more importantly, each of us viewed our current world differently, yet for all us in 1975 Hawaii, there was only one world that we lived in. And perhaps that just wasn't Hawaii, but the entire world, and all of what we heard about the entire world, in places where weren't, but where we were interested.

So it's straightforward to be knowledgeable, at least about certain things, things that interest you. Your immediate environment interests you, and so do certain other things, whether tangible or not. There is generally motivation that we all need to change. So everyone can achieve happiness in life, just by knowing about their worlds and how their worlds interact with other worlds, perhaps wondering about worlds unknown to them. However, there are problems that we experience in interpreting the worlds. There are physiological problems, which arise from age, like loss of memory, and there are all the evils that come into being, fear, guilt, etc., all psychological problems. Most are powerful enough to not only alter our perception, but also limit, or entirely block out, acquiring knowledge. Another limiter is denial, and the most obvious is missing information – one does not have access to most knowledge.

My friend Catherine from San Francisco knew nothing about Hawaii. She was the type that liked to experience things rather than hear about them. Before she came to visit, she was missing information. When she arrived, and it rained on her without her seeing any clouds, she experienced denial; when she went to the fast food restaurant for lunch, she had difficult eating because she lacked the co-ordination to use chop sticks. And she couldn't sleep well in my apartment because she was afraid of lizards. To say the least, my world of Hawaii was not a very pleasant place for Catherine. She found staying with Peggy a bit more pleasant. The absence of the cockroaches pleased Peggy and because of that, Peggy was open to learning more about Hawaii. The absence of knives and forks, and the presence of house lizards didn't please Catherine, so she wasn't open to learning more about Hawaii, in fact she wanted to leave as soon as possible. There is a balance between the pleasing attributes and the displeasing ones that determines the amount of new information that one let's in - there is a range of amount of acquisition, starting with 0 and going to thriving.

At this point, it is clear that "belief" situations are the fodder of sociological and psychological complexity. And speaking of complexity, already we've experienced a headful of thoughts, and I've just started with a few beliefs. How complex is a life full of them? Yet we all have one, and expand it every day. How is this done?

Language

The acquisition of the natural language of a culture comes about by mimicking. I believe that the mimicking comes before meaning for most – have you ever heard a 5 year old repeatedly repeat what you say, without having a clue as to what it means? That the language must provide a clear representation of the world is completely academia, and varies in degree from one natural language to the other. A sparse language, like Native American (Indian), which arises from a sparse environment, has one degree of complexity; while another, like French, has another degree, representing a higher complexity. No matter what the degree, we can disregard the actually mechanics of a language when dealing with meanings. The forming of sentences can be thought of as being done on a syntactical tablet, on your left, and the thoughts of the world they refer to as belonging to another semantic tablet, on your right. There is a good and proper mapping of the tablets, an isomorphism. The right tablet represents the world, and it is to be noted that the culture creates the natural language. So the mimicking must be mapped to the world before it can become a part of the individual's language. Thus, the language is created within the culture; and we find different cultures talk about the same worlds in different ways. The language belongs to the culture and those words and phrases that are limited to a particular culture are ported over to the cultures, intact, while words describing an event common to two cultures will most of the time be different in the languages, but the translations will be precise; as in "rain" in English and "pui" in French. However, many words, and phrases, in one language are not directly translatable into another language, simply because of the language. An example of this is the translation of sentences in English to French involving motion. In other cases there is no translation because the world of one culture is lacking an element in the world of another, as in "chopstick" being carried over from the Chinese word "kuaizi", being imported from the culture – not being part of the original western world.

Atop of any language, and both parts of it, is (a) logic, common to both parts. Along with it comes reasoning.

Logic and Reasoning

From our very early days as babies, we learn logic. We learn it from practical experiences. We learn the most primitive element of logic, called modus ponens; generally referred to as implication. While implication is not the same as causation, we learn about implication initially through causal occurrences of events. Events are instances of life that may, or may not, happen. Thus, one may say that if I touch a hot stove with my bare hand, I will feel pain. Alternately, "touching a hot stove bare handed implies I will feel pain". Here, we use implication in place of casualty. And this type of implication is learned by all of us, by repeated application throughout our early growing up stages, as in the learning to deal with nature, (like heat, cold, dry, wet), with acquaintances (like friend, enemy, authority, control), etc. "Implies" can simply be regarded as the phrase "it follows that". Topically, it is referred to as entails. Classically, and in computer science, is can be replaced by If ..., Then...., as in "If I touch a hot stove bare handed, then I will feel pain", "If I shout at my mother, I will be disciplined", "If I shove a friend, the friend may shove me back; but if I shove an enemy, I might have to fight"; the "then" is often left out. If, at a gathering of employees, I tell the president of the company that he has lost weight, and I hear and see a dead audience, I know better next time around.

Examples of implications that do not involve causality stem from the use of implication in describing logical consequences that are not action-events. Implication is not the same as causality when both statements do not represent (immediate) actions.

1. "Seeing a red balloon in the sky implies the sky was clear enough to see the red balloon"

Seeing the red balloon does not cause the sky to be clear. And the sky being clear does not cause me to see the red balloon. That the sky is clear enough to see the red balloon is not an action-event. If I'm looking into the sky, and there is a red balloon in the sky, and the sky is not clear, and then, the event happens that the sky clears, it can be said that the sky clearing caused me to see the balloon.

Seeing a red balloon in the sky <u>implies</u> many things. One is that I'm looking into the sky, but a more fundamental one is that something, or someone, has <u>caused</u> a red balloon to be in the sky (a child lost control of it on the ground, or it is a weather balloon and was release by weatherperson, or..). The causality has been left out of statement 1.

However,

2. "Seeing a red balloon in the sky implies that I was looking in the sky, and the sky was clear, and a red balloon is in the sky"

takes into account the missing causality in 1. Given any statement like 1., with implication given but casualty missing, a statement like 2. can be found that provides the missing causality, even if we have to use all possible statements that could be made (that is, we may not know what causes the event). Thus the difference between causality and implication can be seen as splitting hairs, but it is still necessary to distinguish between the use of the two words. We can think of implications as strings of sentences that combined via conjunction are a causal statement, the one that includes all of the "it follows that" statements. Given that A implies $B_1, \ldots, B_n$, A implies B_i , so the each B_i should be thought of as just a possible part of the cause of A. Vis a vis for $B_1, \ldots, B_n$ implies A. Care should be taken to exclude from the B_i statements that have no relation to A, but are true nevertheless.. Note that a statement of causality can always be replaced by one of implication, but not visa-versa.

If A implies B, written A→B, indicates that whenever A happens then so does B. If as well as B implies A, there are no occurrences of A without B, then B→A, and thus the two statements A & B are equivalent.

The use of "and", "or", and "not" is again given to us by childhood education and the learning of the language we use and how they relate to the validity (truth or fallacy) of the statements that they are used in.

We mean by the statement *Dan does not smoke*, we can write the sentence more formally as *it is not the case that Dan Smokes*. Equivalently, using ~ for "not", we can write ~ *(Dan Smokes)* to make it a formal, symbol sentence.

As I've said before, it is important to split the situation into two, so to speak, and think of statements in the situation as written in everyday English, and formal symbolic sentences as being their counterparts in a formal language. I further separate the writing of the two systems by putting sentences of the formal language on the left, and statements of the situation on the right. It should be clear that this can be done without ambiguity of which statements on the left are linked to which statements on the left. We will also go further to use the implication symbol "→" on the left linking to "If … Then", or entails, on the right, as in

P → Q ◇◇◇◇◇ *If P then Q* (, or *P entails Q)*

We continue to use A,B,P,Q, and R on both sides to represents sentences on the left and statements on the right.

We use the symbol

~ on the left for the counterpart of ***it is not the case*** on the right
v on the left for the counterpart of ***or*** on the right

& on the left for the counterpart of ***and*** on the right

By validate A, it is meant to show by truth tables that A is true. Recalling truth table construction, we recall that

P	Q	P or Q	P and Q	If P then Q
T	F	T	F	T
T	T	T	T	T
F	F	F	F	T
F	T	T	F	T

P	it is not the case that P
T	F
F	T

Validation is only done on the statements of the situation, the ones on the right. When A is valid, we write $\models$A.

Easy examples

By inference A, it is meant the ability to derive (or prove) a sentence. A sentence A in inferred from a set of sentences K, if starting with K it is possible to apply the rules of logic on the sentences in K to obtain A. The one rule most often use unconsciously is that of detachment, or modus ponens, namely

From P→Q and P, infer Q. This is written formally as (P→Q) & P $\vdash$ Q

where "$\vdash$" is the symbol of deductibility.

Among them are some familiar to you, like

FROM	INFER	
P	~(~P)	double negation
P & (Q v R)	(P & Q) v (P & R)	& distributive
P or (Q v R)	(P v Q) & (P v R)	v distributive
~(P & Q)	~ P v ~Q	De Morgan's Law
~(P v Q)	~P & ~Q	De Morgan's Law
P → Q	~Q → ~P	the Contrapositive

Actually, all the rules correspond to what are termed tautologies, in that they are universally valid – true under any interpretation of the symbols - ie. true in any situation, their truth being evaluated on the linked counterpart statements on the right.

It can be shown that all of these tautologies can be derived from modus ponens, by appropriate substitution, in certain sets of axioms of logic, most profoundly the smallest one discovered by Jan Łukasiewicz; so that it is possible to derive the entirety of true statements given an initial set of sentences and the logic axioms by just using modus ponens and 3 axioms – which are statements taken to be true.

For more info: http://en.wikipedia.org/wiki/Propositional_calculus
or http://www.iep.utm.edu/prop-log/
or Formal Logic for Informal Logicians

However, most of us think using "&" and "v" and "~" and we might find ourselves reversing the idea and converting modus ponens strings into their equivalents using these connectives. Worth remembering is that

P → Q is equivalent to ~(P & ~Q) is equivalent to (~P v Q).

Easy examples:

(P → ~Q) & P → ~Q :	(P → Q) & P → Q	modus ponens
	(P → ~Q) & P → ~Q	substitution of ~Q for Q
(R & (Q V R) → P) → P :	(Q V R) → P	given
	R → P	
	R & (R → P)	R given
	(R → P) & R	commutative
	R	modus ponens

(Q v R) & P → (Q → P) v (R→ P)

Here’s a derivation of ((P → Q) & (Q → R)) → (P → R)

1. P → Q	Premise
2. Q → R	Premise
3. ~P v Q	1 Material Implication
4. (Q → R) v ~P	2 Addition
5. ~P v (Q → R)	4 Commutation
6. P → (Q → R)	5 Material Implication
7. P → (P & (Q → R))	6 Absorption
8. ~P v (P & (Q > R))	7 Material Implication
9. (~P v P) & (~P v (Q > R))	8 Distribution
10. ~P v P	9 Simplification
11. P → P	10 Material Implication
12. ~P→~P	11 Transposition
13. (~P → ~P) & (Q → R)	2,12 Conjunction
14. ~P v R	3,13 Constructive Dilemma
15. P → R	14 Material Implication

Exercises: Translate the following and prove the conclusion, if possible, using inference, and then validate.

1. If Jennifer loves dogs, then she doesn’t have a cat. Jennifer loves dogs. Therefore Jenifer does not have a cat.

2. One or the other likes cats, Manny or Minny. Manny likes dogs. If a person likes dogs, then the person doesn’t like cats. Therefore, Minny likes cats.

3. Patrick has a pet lizard, and Norman has a truck. If a person has a truck, the person can not have a lizard. So Norman has a pet lizard, and a truck.

4. Linda is a BBA student. A person studying for the BBA must take a class in Business Reasoning. Manny is taking a class in business reasoning. Therefore, Manny is a BBA student.

5. Linda is a BBA student, and Vicki is studying business reasoning. It is not the case that, if a person is a not BBA student, then the person must study business reasoning. So Vicki is studying business reasoning and she is a BBA student.

6. If wages are high then prices are high. Wages are high or times are bad. Times are good. Therefore, prices are high.

7. If Jim is at the jail after visiting his sister. His sister was robbed, and Jim has her purse. The robber is at the jail. Jim got his sister’s purse from the robber or Jim is the robber. If

Jim didn't rob his sister then Jim is picking up the purse at the jail. Therefore, Jim robbed his sister.

8. The parts on are on the 4th floor or in the basement. Mr. Jenkins say's they are in both places, but Mary says Mr. Jenkin remembers only when they were in the basement. However, Mr. Jenkins also says Mary always reverses what used to be the case. Therefore, the parts are on the 4th floor.

9. Above I showed that given that $P \rightarrow Q$ & $Q \rightarrow R$, $\vdash$ $P \rightarrow R$.
Use truth tables to show that If P then Q and IF Q then P $\models$ If P then R.

--------- Just write these sentences symbolically.

10. You've sent me emails about nothing every three days. Either quit sending me unimportant emails or unsubscribe me from your service.

11. I promised my son that I would either buy him a fast motorcycle or no motorcycle at all.

Languages and Models

All languages can be looked at as split in two pieces. The syntactic part, and the semantic part; the first having to do with the rules of how sentences are formed, and latter with rules on how to interpret the sentences formed using the rules of the first part. While separate pieces, they are conjoined at the hip and created bi-laterally. However, for everyday thinking it is advantages to keep them as far as part as possible, as the meanings of what we write or say are always at bay with the true meanings of the situation at hand, and not worry about how the two are connected.

There's much to do about how the two parts are connected, and that the grammar of each part matches the grammar of the other. And, in order to avoid a blizzard of unnecessary complexity, we will work within formal languages which give a 1 to 1 correspondence with the English natural language. That way, an every-day situation can be represented in the formal language for discussion and analysis. For those interested in going to the deep into the analogous natural language counterparts of this discussion, I invite you to try and good book about Montague Semantics. For one, _________

Montague is the responsible party for the extension of formal language linguistics to natural language linguistics. I will point out along our way any Montague counterparts that can be seen at a glance.

It's convenient to think of two tablets, one on the right contains the set of natural English statements, S, of the account, or model, of the situation. The atoms of S are the literals use to make the axiomatic statements of the situation. In S, a well-forming procedure can be performed to obtain every valid consequence of axioms, so we can think of every statement as belonging to the model; and there-in each is labeled with T or F using the truth table logic. However, for the most part, we view the model with just premise statements listed, knowing that the sets of statements mentioned can be generated if need be.

The tablet on the left contains the set, D, of formal, symbolic sentences that describe the situation, S. The atoms of S are P,Q, ... The axioms are the sentences that describe the situation. In D, a well-forming procedure can be performed to obtain every derivable consequence of the axioms. However, for the most part, we view the model with just atoms listed, knowing that the sets of statements mentioned can be generated if need be. Classically, an interpretation of a logic system of propositions maps each proposition into {T,F} so that the situation has no structure associated with it -- it relies on the propositions for the structure.

Instead of that, here we provide a mapping of D into S in such a manner that each statement of S has a unique representation as a sentence in D. The mechanics of coordinating the mapping and logic is beyond the scope of this article and those interested should read from the more technical materials listed in the references, Introduction to Montague Semantics for a softer (however, sloppy) treatment.

For instance, consider a small situation “Dan loves Peggy” entails “It is not the case that Peggy smokes”.

MAP:	**Literal** into	**Atom**
	P	Dan loves Peggy
	Q	Peggy Smokes

Map: **D** **S**

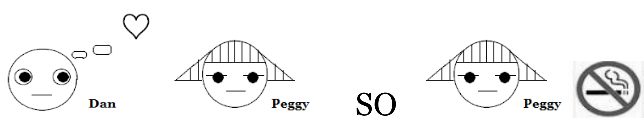
$P \rightarrow \sim Q$ If Dan loves Peggy then it’s not the case that Peggy smokes T

S’

$P \rightarrow \sim Q$ [It’s not the case that D Loves Peggy] o**r** [It’s not the case that Peggy smokes] T

S”

$P \rightarrow \sim Q$ a picture, reading, film, imagination, etc.



T

Note that we haven’t assigned the atoms any T or F values. Our only premise in S is that <u>If Dan loves Peggy then it’s not the case that Peggy smokes</u>. What can we conclude in S. We know ~P or Q.

There can be many representations of the situation, S, S’,..., each isomorphic to S.

We focus on only one S, and its description, D, even though there are many possible meaningful and elaborate models involving pictures, film, or even just imagination. Actually, using a model that lies in your imagination is common, and often called an interpretation, or perspective, of the situation. See **Tom Burke.**

In S and S’ I have use natural language statements. In fact, both S and S’ could be thought of as descriptions of S”. The “glue” of the language system is the back and forth binding of the logic on both the model and the situation, including the valuation rules – which simply repeat our we” feeling about truth tables. And actually, any description D can be made into a model of D by the appropriate mapping, keep the symbols actually the same. (This in a more general and advanced sense, is called forcing).

Let ξ denote the injection which maps sentences of D into the statements of S. Since a model can be bigger than a minimal model, ξ is not isomorphic. Thinking in terms of situations S, or fragments, of the world, S never is a large model. Since most worlds are infinite, one would be hard pressed to be working in a natural language (or representative of one) with them, as is done with formal languages and models comprised of set theoretic structure designed to model the language needed to express mathematical ideas.

Since, most generally, the sentences are the same on each side, the mapping symbol is omitted. However, we could write

ξ (Dan loves Peggy) = [Dan loves Peggy], etc.

The left can be thought of as a theory, while the right as the actuality. The left and right physical placement helps separate inference from validity, and syntax from semantics. The inferential engine is used on the left tablet to derive sentences that are logical consequences of the axioms+logic axioms+modus ponens; and the validity (or truth) engine is used on the right tablet to show that certain sentences are valid (or true) consequences of premises + truth table evaluation. That is, we infer on the left, and validate on the right.

In general, a situation's description, or theory D (in a logic calculus) is called complete when any true statement, any statement valid in every interpretation (model) of it, can be derived in it. A statement, φ, is true in the interpretation, S, is written $S \models \varphi$.

So D is complete description of S with mapping ξ from D onto S, whenever, for every φ in S, if $S \models \varphi$ then $D \vdash \xi^{-1} (\varphi)$.

That for every φ in D, if $D \vdash \varphi$ then $S \models \xi (\varphi)$ is termed soundness, and every theory is sound wrt its models.

See the article by Neil Tennant on the completeness of propositional calculus at the following link neil tennant CPL.

For a situation the question of being able to decide on the validity (in the interpretation) of any statement made in the language describing the situation by some algorithmic procedure applied to the language describing the situation. Completeness and decidability aren't the same and if interested, you can read the article by Curtis Brown on the decidability of propositional calculus at the following link curtis brown DPL . In higher order calculus systems, there are decidable theories which are not complete, and vice versa. The next higher order calculus we shall talk about is the predicate calculus along with quantification and what's commonly called first order logic (FOL). We will take our situation Hawaii, 1975 as the model, and create the theory (description) in the syntax of necessary to talk about the model in FOL.

But first, let's take a look at Hawaii, 1975 theory in just the low level of propositional calculus. And we'll mix valid with derivable, as for the most part, this is the way people think.

Lizards eating with chopsticks, rain falling everywhere, and Catherine wants to leave.

MAP:	Axiom into	Atom
	C	Clouds in the sky
	R	Rain

etc..

Thus, in Hawaii, 1975,

"There are no clouds in the sky implies it will not rain", ~C → ~R, is not true but is the same as saying the contrapositive, that "It will rain implies there are clouds in the sky",R → C. Thus the statement "There can be no clouds in the sky, and it can still be raining", ~C & R, is true. Given that ~C & R, infer that ~~(R&~C) so that ~(R → C).

In Hawaii, we have listed sentences, in D, that are true in the model, so derivable in the, in the description of the situation. The negations of these sentences are each false. So, using the sentences given, and their negations, we have an interpretation, S, of them taken altogether. Since this is a true to life account of a situation, we know that there are no two statements in S that contradict each other. That is, no statement φ, in S, for which φ and its negation are both true. Thus, for no statement φ, in D, is both φ and ~φ derivable. Such a set of sentences are said to be consistent, and have a model.

Putting the idea in reverse, a set of statements has a model if the set is consistent. That is, we have gotten the statements from the model, Hawaii 1975. In reverse, one could write down any set of statements and, if there were consistent, a φ and ~ φ do not belong to the set, then there is a model for them. A model for them is often taken to be the forced model, the one in which a symbolic form of the statements is created, and then the symbols are considered as the elements of the model.

In our model for Hawaii, 1975, the statements in S that Lizards eat cockroaches; fast-food is served with chopsticks; rain falls from the sky when clouds are not seen; Peggy is in denial, …, etc. are true. They map to the axioms of our description.

Peggy lives in a good area causes Peggy to be in denial so that

1) Peggy lives in a good area → Peggy is in denial (of cockroaches)

2) Bill is accustomed to cockroaches → Bill doesn't have lizards and Bill can eat with chopsticks

Actually, one might balk at 2) but since Bill doesn't have lizards and Bill can eat with chopsticks is true, the sentence is derivable.

3) Dan has lizards → Catherine wants to leave

Because Catherine wants to leave, 3) is derivable. But the reason that Catherine wants to leave is that she doesn't like cockroaches and can't eat with chopsticks. Again, we could say that what might cause a person to leave is the dislike of cockroaches and also the inability to eat with chopsticks, as in 4)

4) Catherine doesn't like lizards and she can't eat with chopsticks → she wants to leave.

However, it could be that

5) Catherine doesn't like lizards or she can't eat with chopsticks → she wants to leave.

where either of the two conditions causes the need to depart.

However, only 4) is given.

If we also knew that Catherine wants to stay with Dan, then along with 4), we know that Catherine leaves.

At any rate, if Peggy visits Dan or Bill enough, Peggy will not be in denial, and if 5) holds as a casual implication, she may depart. I leave the casual justifications of repeated visits to both me and Bill to your imagination; along with the differences of the effect of the visit to each of us.

Exercise: What causal justifications for Peggy leaving do you assign to repeated visits to Bill? Repeated visits to Dan?

This can be formulated as

6) ((If Peggy repeatedly visits Dan or repeatedly visits Bill) and (she doesn't like cockroaches)) → she wants to leave

$$((A \text{ or } B) \,\&\, C) \rightarrow E) \;=\; \sim((A \text{ or } B) \,\&\, C)) \text{ or } \sim E = \sim((A \,\&\, C) \text{ or } (B \,\&\, C)) \text{ or } \sim E$$
$$= ((\sim A \text{ or } \sim C) \text{ and } (\sim B \text{ or } \sim C)) \text{ or } \sim E = (\sim A \text{ or } \sim C \text{ or } \sim E) \text{ and } (\sim B \text{ or } \sim C \text{ or } \sim E)$$

where A is that Peggy repeatedly visits Dan; C is that Peggy doesn't like cockroaches; B is that Peggy repeatedly visits Bill; E is that Peggy wants to leave

In a theory, sentences can be expressed in the form of a conjunction of clauses, each clause being a disjunction of literals, the above is an example. In fact, all the axiomatic sentences can be expressed in a single sentence of this form. However, this doesn't generally give us more or easier to handle information about the situation. However, replacing implementations with disjunctions is often helpful in remembering which literal must be true, since for every literal in a clause, at least one must be true to make the whole conjunction true, remembering that A_1 & ... & A_n is true when and only when all the A_i are true. In other words, once in clause normal form, CNF, a sentence can be singled out to be false by finding a clause containing literals that are all false.

Looking back at 6)

(~A or ~C or ~E) and (~B or ~C or ~E)

Since C is taken as an axiom, C is true and ~C is false, so the above becomes

(~A or ~E) and (~B or ~E)

so that Peggy doesn't want to leave only when both ~A and ~B.

Exercise: Put 4) into CNF.

Because we have restricted our language to the simple set of axioms we are limited to talking about other statements of the situation. In general, sets of consistent sentences are not decidable. Also, descriptions framed in propositions are not complete, in a more fundamental way than the completeness talked about earlier, unless we capture all of the necessary statements as the axioms. In Hawaii,1975 we must assume that lizards do not eat with chopsticks, Peggy does not eat cockroaches, etc. None of these statements is derivable from the description of the situation, simple because no axioms refer to them. That is, the statement

7) Cockroaches sometimes eat with chopsticks

cannot be evaluated T,F because it does not follow from the atomic sentences. However, the theory describes situations in which many statements cannot be talked about. The axioms are describing a situation that is not the "compete" situation. Needed is another valuation, to signify that the truth or falsity of the statement is unknown. We use N for this and the logic becomes three valued, T, F, N.

The new truth tables can be generated by easily: T or N = T; F or N = F; N or N = N; ~N = T or F; T & N = N; N & N = N; etc.

Exercise: Create the T,F,N truth table for P → Q.

Thus a three valued logic is needed for the "total completeness" of the situation. More than just having a third evaluation, N, is to have a forth, representing necessity. And further, we can attach a probability function to statements in the model. By doing this, we can talk replace false statement with a probability of 0, and with probability 1.

What probability would you assign to 7). How about the probability of Bill eating cockroaches?

Recall that a likelihood, or probability, function _P, maps into [0,1].

…

Such a system is called a probability calculus or probability logic. A theory framed in it can be thought of a probabilistically complete system.

Exercises:
1. What is the probability of a certain stock selling at $3 dips to $2.5 in the next 3 months if it is know that the probability of it losing 1% of it's value in a months time is .3.

2. Given that stock A and Stock B are correlated …

Making the situation complete by introducing predicates.

The next level of dealing with situations is to extend them into first order logic where we can pick up most of the necessary atoms by using predicates. Situations are called theories, and interpretations are called models. Models are thought of a set theoretic structures <A,R,…> where A is a set of individuals and R is a set of relations individuals.

In a more general since, a model is a structure ….., and a theory is a set of sentences in a language (with a propositional logic) that are satisfied by the model. For example, the theory of a group in mathematics is the set of axioms for a group; and all the set theoretic structures for which all the axioms of the group hold (are true) are models of the theory. This pierces the most beautiful, yet most abstract, part of mathematics called model theory. It is quit alarming that most people have never heard of this, however, many have heard of models, and that they are just the reverse of what is thought of as in model theory. That is, a set of sentences, usually equations, is taken to be a model of some physical phenomena, structure. In this case, the axioms for the group of rotations of a triangle would be taken as a model of a group. More likely, the sentences describing a set of linear equations and an objective function to be maximized on the feasible region given by the set is termed the model, and elements that are generated by the sentences are not termed, or perhaps termed as the situation, and perhaps all such situations, or problems, are termed as the theory of linear programming. The switch-around is never a problem, and most high-on in the academic field are acquainted with that differences in terminology and have no difficulty in both dealing with them and presenting in a manner that respects each one.

Kripke worlds.

“Dan likes Peggy more than Catherine”, “Bill will go to New York next week”, “The Sahara dessert is not France”, can all be taken as statements in the situation Hawaii, 1975. However, you may see some differences in these three. In the first, the statement is about the situation. In the second, the statement involves a place, New York, that is external to the “current” situation, and in the third, the whole statement is external to the situation.

When a sentence is true in a particular universe of discourse, we term the universe a model of the sentence. Thus Hawaii, 1975 is a model of 1., 2., and 3. And if we go back to all the sentences from the previous discussion about Peggy, Bill, Catherine and Dan, we see that Hawaii, 1975 is a model for all of them. Technically speaking, we call a set of sentences that are true in a model M, a situation that is satisfied by M. M might need to be expanded to have a satisfactory representation, and so the atoms and language that we use to describe M must be expanded as well, even when we include modal logic.

How do antique dealers recognize value; how does a nose distinguish between senses. Note that a professional perfumer “nose” can distinguish between thousands of scents, some remarkably close in chemical structure.

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03: Mathematical logic and foundations

Introduction

Mathematical Logic is the study of the processes used in mathematical deduction.

The subject has origins in philosophy, and indeed it is only by nonmathematical argument that one can show the usual rules for inference and deduction (law of excluded middle; cut rule; etc.) are valid. It is also a legacy from philosophy that we can distinguish *semantic* reasoning ("what is true?") from *syntactic reasoning* ("what can be shown?"). The first leads to Model Theory, the second, to Proof Theory.

Students encounter elementary (sentential) logic early in their mathematical training. This includes techniques using truth tables, symbolic logic with only "and", "or", and "not" in the language, and various equivalences among methods of proof (e.g. proof by contradiction is a proof of the contrapositive). This material includes somewhat deeper results such as the existence of disjunctive normal forms for statements. Also fairly straightforward is elementary first-order logic, which adds quantifiers ("for all" and "there exists") to the language. The corresponding normal form is prenex normal form. In second-order logic, the quantifiers are allowed to apply to relations and functions -- to subsets as well as elements of a set. (For example, the well-ordering axiom of the integers is a second-order statement).

In Model Theory, one asks for a description of the structures which satisfy some set of axioms (e.g. the axioms for a group or topological space). In first-order languages, the results are striking. For example, the Löwenheim-Skolem-Tarski Theorem asserts that if there are any models, then there are models of every infinite cardinality, unless there is a (finite) upper bound on the cardinalities of the models (assuming the language is countable). The compactness theorem asserts that a model exists for infinite sets of axioms, as long as every finite set of them has a model. (This is true for axioms in sentential logic, and true but deeper in first-order logic; it fails for second-order logic). For example, since all finite maps are 4-colorable, the same is true of infinite maps.

Proof theory is the study of certain kinds of symbol manipulation. Begin with a language -- a set of symbols and a set (the "syntax") of strings of those symbols; elements of this set are "formulas" in the language. A collection T of these formulas is called a "theory" (or more precisely the "axioms" for the theory); for example the theory of groups is expressed by a few axioms in a language which include symbols "=", "*", "x1", "x2", ... as well as symbols "\arrow", "\wedge", and "\forall". We are interested in the "theorems" of T, which is the smallest set S of formulas which includes T and is closed under certain operations (the "rules of inference"), such as modus ponens (if both "A \arrow B" and "A" are in S, then "B" must also be in S).

So how can we characterize the set of theorems for the theory? The theorems are defined in a purely procedural way, yet they should be related to those statements which are (semantically) "true", that is, statements which are valid in every model of those axioms. With a suitable (and reasonably natural) set of rules of inference, the two notions coincide for any theory in first-order logic: the Soundness Theorem assures that what is provable is true, and the Completeness Theorem assures that what is true is provable. It follows that the set of true first-order statements is effectively enumerable, and decidable: one can deduce in a finite number of steps whether or not such a statement follows from the axioms. So, for example, one could make a countable list of all statements which are true for all groups.

In some cases, even more is true: a theory is *complete* if all its models are elementarily equivalent (the same sentences are true in each). In that case, any statement in that language is decidable: it's either true (in all models) or it's false (in all models). Among examples of this are the theory of algebraically closed fields of characteristic zero (Los-Vaught) and the theory of the real field R (Tarski). Any theory T is contained in a complete theory T' using the same language (Lindenbaum); a key issue is just how "bad" T' might be. A famous example is the theory of arithmetic: Using symbols 0,+,*,^,<, and S (successor), let T be decidable set of axioms (e.g. a finite set!) which are valid in the set of natural numbers. Gödel showed T cannot be complete, that is, there are statements in the language which are also valid in the set of natural numbers but which are not theorems of T; equivalently, any complete theory T' containing T (e.g. the set of all statements which are valid in the natural numbers) cannot be decidable -- loosely, T' must be so complicated that we cannot even decide whether a given statement is in that collection of axioms!

The topic of being able to recognize membership in a set is the province of Recursion Theory. Here one asks what is calculable in a finite number of steps. To the extent that one "calculates" a proof of a theorem, this is the question of decidability in proof theory. But the term is more commonly used specifically to mean the calculation of values of natural number functions; this can be shown to embed larger questions such as the satisfiability of a predicate. Here one uses Church's thesis, the convention that "calculable" means what is formally defined (inductively) as recursive. Recursive functions can be described as those which may be computed in finite steps by a simple machine (a Turing machine); a recursive predicate is one whose truth value (0 or 1, say) can be computed in a finite number of steps. (A predicate has the weaker property of being "recursively enumerable" if an algorithm exists which will in a finite number of steps return a value of 1 if the predicate hold; since the amount of steps is not bounded in advance, lack of a conclusion from the algorithm at any time may or may not mean the predicate holds.) Hilbert's tenth problem was to decide if the solvability of Diophantine equations was decidable; Matijasevic (1971) showed it is not, that is, one cannot describe a Turing machine which computes whether each Diophantine equation is solvable or not.

Algebraic Logic studies logical systems via associated algebraic structures. In particular, this is a convenient setting for the study of many-valued logics (more truth values than just "true" or "false"). Just as ordinary logic may be studies with Boolean algebras one may formalize the calculi with many-valued logics using other algebraic systems. These include the algebras of Post, Lukasiewicz, Heyting, etc. This topic leads to a study of abstract algebra systems (lattices, filters, and so on).

In the preceding discussion it has been tacitly assumed that the idea of a "set" is unambiguous. Indeed this is not true, as was noticed a century ago. Much of mathematical logic was developed in response to the questions surrounding the axiomatization of set theory. From this developed the constructions and investigations of very large infinite sets. Descriptive set theory considers various classifications of sets. Cardinal arithmetic considers the "size" of various sets; ordinal arithmetic refines this to the "ordering" of sets. Fuzzy set theory replaces the yes/no statement of set membership with a qualitative predicate.

History

Elementary logic has been studied since ancient times, in part through the analysis of paradoxes, and as a part of rhetoric. Late in the nineteenth century Frege and others attempted to formalize mathematics and the laws of deduction. At the turn of the 20th century a debate arose regarding the legitimacy of non-constructive proofs; Hilbert suggested a program demonstrating the possibility of securing mathematics onto a formal foundation. (Whitehead and Russell (1910-1913) actually tried it.) The necessary groundwork in logic, outlined above, was laid during the 1920s and 1930s, which is when some of the most paradoxical results were obtained. The implicit dependence on set theory and the inability to determine a decidable set of first-order axioms for set theory have caused considerable consternation among mathematicians, particularly those confronted with difficulties associated with the axiom of choice. Notable among postwar developments is Robinson's application of model theory to develop Nonstandard analysis and use it as a framework for ordinary calculus. Many undecidability results appeared in the 1960s and 1970s with applications in traditional branches of algebra. With the development of computer science during later decades, many topics in recursion theory and proof theory were developed from the perspective of the theory of algorithms.

See also "Perspectives on the history of mathematical logic", edited by Thomas Drucker. Birkhäuser Boston, Inc., Boston, MA, 1991. 195 pp. ISBN 0-8176-3444-4 MR94b:03007

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